Combinational Complexity Measures as a Function of Fan-Out

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If $C_s(f_1, \cdots, f_L)$ is the fan-out s combinational complexity of the functions $f_1, f_2, \cdots, f_L$ with respect to straight-line algorithms (or combinational machines) of fan-in r, then it is shown that

$$C_\infty(f_1, \cdots, f_L) \leqq C_s(f_1, \cdots, f_L)$$

$$\leqq \left(\frac{d(r-1)}{s-1} + 1 \right) C_\infty(f_1, \cdots, f_L) + \frac{d}{s-1}(L-N)$$

where N is the number of variables on which $f_1, \cdots, f_L$ depend and $d = C_s(I)$ where I is the identity function in one variable. Thus, a well-designed combinational machine or algorithm will not have a fan-out which is more than several times its fan-in.

I. Introduction

In this paper we develop bounds on the fan-out s combinational complexity of functions. These bounds show that the combinational complexity of functions has a weak dependence on fan-out when $s \geqq 2$.

II. Bounds on Combinational Complexity

Before we develop the promised bounds, we state the following definitions which are needed in the sequel.

DEFINITION 1. *Let Ω be a set of functions over the set Σ, such that if $h_i : \epsilon \, \Omega$, then $h_i : \Sigma^{n_i} \to \Sigma$. Let*

$$\Gamma = \Sigma \cup \{X_1, X_2, \cdots, X_N\}$$

Then, an (Ω, Γ) algorithm (or "straight-line" algorithm) is a K-tuple $\beta = (\beta_1, \beta_2, \cdots, \beta_K)$ where either $\beta_k \in \Gamma$ or $\beta_k = (h_i; k_1, k_2, \cdots, k_{n_i})$, $h_i \in \Omega$, $1 \leqq k_t < k$. The set of functions $(\bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_K)$ is associated with β where $\bar{\beta}_k = \beta_k$ if $\beta_k \in \Gamma$ and $\bar{\beta}_k = h_i(\bar{\beta}_{k_1}, \cdots, \bar{\beta}_{k_{n_i}})$ if

$$\beta_k = (h_i; k_1, k_2, \cdots, k_{n_i})$$

An algorithm β is said to compute the functions

$$f_l : \Sigma^{m_l} \to \Sigma, \qquad m_l \leq N, \qquad 1 \leq l \leq L$$

if there exist $\beta_{m_1}, \cdots, \beta_{m_L}$ such that $f_l = \beta_{m_l}$

The *fan-in* of Ω is

$$r = \max_i n_i$$

where $h_i : \epsilon\, \Omega$, $h_i : \Sigma^{n_i} \to \Sigma$. If β computes $f_1, f_2, \cdots, f_L$ where $f_l = \beta_{m_l}$, $1 \leq l \leq L$, let γ_i the number of steps of β which use β_i, if $\beta_i \notin \Sigma$, and $\gamma_i = 0$, $\beta_i \in \Sigma$ and let $\theta_i = \gamma_i$, $i \neq m_1, m_2, \cdots, m_L$ and $\theta_i = \gamma_i + 1$ otherwise. Then, the *fan-out* of β is

$$s = \max_i \theta_i$$

DEFINITION 2. *The combinational complexity with fan-out s of*

$$f_l : \Sigma^{m_l} \to \Sigma, \qquad 1 \leq l \leq L, \qquad C_s(f_1, \cdots, f_L)$$

is the smallest number of steps $\beta_k \notin \Gamma$ of any (Ω, Γ) algorithm which computes these functions, if one such exists; otherwise $C_s(f_1, \cdots, f_L)$ is ∞. Associated with any (Ω, Γ) algorithm is a graph G in which vertices correspond to steps of the algorithm and edges are directed and ordered from vertices corresponding to $\beta_{k_1}, \cdots, \beta_{k_{n_i}}$ to the vertex corresponding to β_k if $\beta_k = (h_i; \beta_{k_1}, \cdots, \beta_{k_{n_i}})$. Vertices corresponding to steps $\beta_k \in \Gamma$ are called source vertices.

Combinational machines are circuits which correspond to the graphs of (Ω, Γ) algorithms in which $\Sigma = \{0, 1\}$ and Ω is a set of Boolean functions; thus, there is an equivalence between combinational machines and straight-line algorithms. These algorithms are called "straight-line" because they do not permit loops or conditional branching. We now state the principal result of this article.

THEOREM. *Let $f_1, \cdots, f_L$ be distinct functions over Σ which depend on N variables. Let Ω have fan-in r and let it be such that an (Ω, Γ) algorithm exists for the identity function I in one variable. Then*

$$C_\infty(f_1, \cdots, f_2) \leq C_s(f_1, \cdots, f_L)$$

$$\leq \left(\frac{d(r-1)}{s-1} + 1 \right)$$

$$\times C_\infty(f_1, \cdots, f_L) + \frac{d}{s-1}(L-N)$$

where $d = C_s(I)$.

Proof. Let β be a straight-line algorithm with fan-out s which computes $f_1, \cdots, f_L$ with $C_s(f_1, \cdots, f_L)$ operations. The directed graph of β has N source vertices and L vertices identified with the distinct functions $f_1, \cdots, f_L$. To the graph G of β add L vertices with edges directed into them from the vertices identified with these functions. The number of edges incident upon vertices in this new graph G' is at most $rC_\infty(f_1, \cdots, f_L) + L$ since each of the original vertices has at most r edges directed into them. Thus, if θ_i edges are directed away from the ith vertex of G' then

$$\sum_i \theta_i \leq rC_\infty(f_1, \cdots, f_L) + L$$

where the sum is over all vertices except those associated with constants.

Since Ω is complete, the identity function on one variable $I(x)$ can be constructed with some number, e.g., d, of elements from it with fan-out s. For each i, if the ith vertex of the graph G' has θ_i edges directed away from it, we can add $h(\theta_i, s)$ copies of the algorithm realizing $I(x)$ to produce a graph G'' which has fan-out s. Here

$$h(\theta_i, s) \leq \frac{\theta_i - 1}{s - 1}$$

so the number of elements in G'' is bounded above by

$$\frac{d}{s-1} \sum_i (\theta_i - 1) + C_\infty$$

where the sum on i is taken over all vertices of G including all source vertices other than those associated with constants. Since $C_s(f_1, \cdots, f_L)$ is the minimum number of operations required to realize $f_1, \cdots, f_L$ with fan-out s, it follows that

$$C_s(f_1, \cdots, f_L) \leq \frac{d}{s-1}(rC_\infty(f_1, \cdots, f_L)$$

$$+ L - C_\infty - N) + C_\infty$$

The left-hand equality of the theorem follows since $C_s(f_1, \cdots, f_L)$ is a non-increasing function of s. QED.

The significance of this result is that all of the complexity measures $C_2, C_3, \cdots, C_\infty$, differ by at most a constant. Also, C_s approaches C_∞ with increasing s when r is fixed. For many sets Ω, $d = 1$; for example, this is true for the set of addition, subtraction, multiplication and divi-

sion over the reals and the set of AND, OR and NOT over the set $\{0, 1\}$. However, $d = 2$ for the set Ω containing only NAND over $\{0, 1\}$.

The combinational complexity of a function with fan-out 1, C_1, can differ substantially from its combinational complexity with unlimited fan-out. Subbotovskaya (Ref. 1) has shown that the Boolean function $f(x_1, \cdots, x_N) = X_1 \oplus \cdots \oplus X_N$ where $\oplus$ denotes the EXCLUSIVE OR has $C_1(f) > a_1 N^{3/2}$ for some constant a_1 when Ω consists of AND, OR and NOT and $C_\infty(f) < a_2 N$ for some other constant a_2.

Reference

1. Subbotovskaya, B. A., "Realizations of Linear Functions by Formulas Using $\wedge$, &, ¯," *Sov. Math. Dokl.*, Vol. 2, 1961.